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Decidability in Orthomodular Lattices

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We discuss the possibility of automatic simplification of formulas in orthomodular lattices. We describe the principles of a program which decides the validity of equalities and inequalities, as well as implications between them and other important relations significant in quantum mechanics.

KEY WORDS: orthomodular lattice; free lattice; ideal; congruence; implication; equivalence.

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1. MOTIVATION

Orthomodular lattices (OMLs) have been introduced in the early works by Birkhoff and von Neumann (1936) as a mathematical structure allowing to describe events in a quantum mechanical system (Gudder, 1988). OMLs are not distributive in general. This causes problems with evaluation of more complex expressions, because these cannot be transformed to a unique normal form. Therefore there are even doubts about the solvability of the word problem in orthomodular lattices (Herrmann, 1987; Kalmbach, 1986). As a substitute for distributivity, several tools were designed, e.g., the Foulis–Holland theorem (Foulis, 1962; Holland, 1963). Among them, the *focusing technique* due to Greechie (1977) seems to be the most general. It was used in most of algebraic results in OMLs (Gudder, 1988; Kalmbach, 1983). It allows to use distributivity under some restrictions; in particular, it cannot be applied to expressions containing a variable and its orthocomplement. An alternative tool has been proposed in Navara (1997). It uses the fact that the free orthomodular lattice with two free generators is finite. Thus all

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OML formulas in two variables can be expressed in a unique way. This procedure has been implemented by Megill and Pavičić (2001) and later by one of the authors (Hyčko, 1987). Here we describe an improved computer program for this purpose which allows not only to decide the validity of equalities and inequalities and also formulas composed from them by the use of (classical) logical connectives. We clarify its principles and also limitations of this technique. We compare it to focusing.

2. BASIC NOTIONS AND CURRENT STATE

An *orthomodular lattice* is an algebra $(L, \wedge, \vee, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, $'$ (*orthocomplementation*) is its antiisomorphism and the following identities hold for all $a, b \in L$: $a \wedge a' = 0$, $a \vee b = a \vee \big(a' \wedge (a \vee b)\big)$ (the latter identity is called the *orthomodular law*; see Beran (1984); Kalmbach (1983) for its equivalent formulations). Orthomodular lattices form an algebraic basis for the description of event structures of quantum mechanical systems. For basics about orthomodular lattices, we refer to Beran (1984); Gudder (1988); Kalmbach (1983). Throughout this paper, L denotes an OML (we use this abbreviated notation instead of $(L, \wedge, \vee, ', 0, 1)$).

We distinguish a *sublattice* of an OML, which is a subset closed under the join and meet, and a *sub-orthomodular lattice* (*sub-OML*) which is closed under the join, meet, and orthocomplementation. A sub-OML which is a Boolean algebra (with the operations inherited from the OML) is called a *Boolean subalgebra* of an OML. Two elements a, b of an OML *commute* (are *compatible*), in symbols aCb, if they are contained in a Boolean subalgebra. There are many equivalent formulations of this property, e.g., the equalities $a = (a \wedge b) \vee (a \wedge b')$ or $c = 1$, where $c = (a \wedge b) \vee (a \wedge b') \vee (a' \wedge b) \vee (a' \wedge b')$ is the *lower commutator* of a, b. An element of an OML is called *central* if it commutes with all other elements. The set of all central elements of an OML L is called the *center* of L and is denoted by $C(L)$. For elements $a, b \in L$ such that $a \leq b$ we define the *interval* $[a, b] = \{x \in L \mid a \leq x \leq b\}$.

The modular ortholattice MO2 is the OML $\{0, 1, x, y, x', y'\}$ whose elements satisfy $u \wedge v = 0$, $u \vee v = 1$ for all $u \in \{x, x'\}$, $v \in \{y, y'\}$. Its center is $\{0, 1\}$.

The Greechie focusing technique (Greechie, 1977, 1979) can be reformulated as follows:

Theorem 2.1. *Let L be an OML and $a_i, b_i \in L$ $(i = 1, \ldots, n)$ be elements such that*

$$\forall x, y \in \{a_i, b_i \mid i = 1, \ldots, n\} : (x\mathrm{C}y \text{ unless } \{x, y\} = \{a_i, b_i\} \text{ for some } i)\,.$$

Then the sublattice *of L generated by $\{a_i, b_i \mid i = 1, \ldots, n\}$ is distributive.*

As a corollary, we may use distributivity for any lattice polynomial in variables a_i, b_i $(i = 1, \ldots, n)$, but the use of orthocomplementation has to be avoided. As the commutation is preserved by taking orthocomplements, focusing can be generalized to the case when some of the variables a_i, b_i $(i = 1, \ldots, n)$ are replaced by their complements. However, it is not allowed to combine a variable and its orthocomplement during one application of focusing.

The approach of Navara (1997) is based on the use of the free orthomodular lattice with two free generators a, b, denoted by $\mathrm{F}(a, b)$.

Proposition 2.1. (Beran, 1984; Kalmbach, 1983) *The free OML* $\mathrm{F}(a, b)$ *with two free generators* a, b *is isomorphic to the product of the Boolean algebra* $2^4 \cong [0, c]$ *(with atoms* $a \wedge b, a \wedge b', a' \wedge b, a' \wedge b'$*) and the modular ortholattice* MO2 $\cong [0, c]$ *(with atoms* $a \wedge c'$, $a' \wedge c'$, $b \wedge c'$, $b' \wedge c'$*), where* c *is the lower commutator of* a, b*; the corresponding isomorphism is* $h: x \mapsto (x \wedge c, x \wedge c')$. *The OML* $\mathrm{F}(a, b)$ *has* $2^4 \times 6 = 96$ *elements, including eight atoms (listed above). Its center is the Boolean algebra with five atoms* $a \wedge b$, $a \wedge b'$, $a' \wedge b$, $a' \wedge b'$, c'. *An element* $x \in \mathrm{F}(a, b)$ *is central if and only if* $x \wedge c' \in \{0, c'\}$*, i.e., the* MO2 *part of* x *is not an atom.*

The complete list of the elements of $\mathrm{F}(a, b)$ is presented in Beran (1984) together with their unique codes (called *Beran's numbers* in Megill and Pavičić, 2001, and subsequent papers).[4] Any formula composed of the elements of $\mathrm{F}(a, b)$ is equivalent to a unique element of $\mathrm{F}(a, b)$; as this OML is finite, the whole computation can be made automatically, e.g., by the use of a computer program. In Navara (1997), a graphical representation of the elements of $\mathrm{F}(a, b)$ was suggested; it allows to memorize these elements and also simplifies the operations, performing them independently on the Boolean factor and on the MO2 factor. Later on, Megill implemented this idea in a computer program[5] which, given a formula in two variables, returns the Beran's number of the corresponding element of $\mathrm{F}(a, b)$. This program was presented in Megill and Pavičić (2001) and extensively used in Megill and Pavičić (2003a,b) and other papers. As the authors paid special attention to quantum implications, they adapted the program particularly for testing hypotheses about implications. For instance, they included operators that might be substituted by *any* quantum implication. Using this tool, they have found formulas which have a given meaning independently of the choice of the implication. Such results could be hardly obtained without computer support.

Recently a new implementation of the method of Navara (1997) has been done in Hyčko (1987). Besides the Beran's numbers, its output contains also the graphical representation according to Navara (1997) and the corresponding TEX

[4] The table of Beran's numbers can be found at http://cmp.felk.cvut.cz/∼navara/FOML/.

[5] http://us.metamath.org/downloads/quantum-logic.tar.gz.

macro for its typesetting.[6] It is not limited to simplification of formulas and testing equalities. It allows to answer more complicated questions as we shall describe in the sequel. Besides, it admits to introduce further variables, provided that they commute with all other variables. This covers many algebraic results found in the literature and allows to verify them by a simple question to the program. Here we present its updated version and simplified proofs of its principle. Then we compare its power to that of the focusing technique.

2.1. Congruences and Ideals in OMLs

Let us recall some basic properties of finite orthomodular lattices, with special attention to the free OML F(a, b) with two free generators. We use the notation and results from Navara (1997).

Definition 2.3. A *congruence* in an OML L is an equivalence relation $\psi \subseteq L^2$ preserving the operations $\wedge, \vee, '$ (thus also 0, 1). Its kernel is $\ker \psi = \{x \in L \mid (x, 0) \in \psi\}$.

The set of all congruences of an OML L with the ordering by inclusion forms a complete distributive lattice denoted here by Con(L). Each congruence can be uniquely determined by its kernel (Birkhoff, 1973; Kalmbach, 1983). Kernels of congruences in OMLs are particular ideals called *p-ideals*. In finite OMLs this notion may be simplified as follows (see Dorfer, 2001, Proposition 2.6; Kalmbach, 1983):

Proposition 2.4. *Let L be a finite OML and ψ be a congruence on L. Then the kernel of ψ has a greatest element, denoted by w_ψ (thus* $\ker \psi = [0, w_\psi]$ *is a principal ideal); moreover, w_ψ is central.*

Proposition 2.5. (Kalmbach, 1983, Section 6, Theorem 10) *Let L be a finite OML. The mapping which maps each congruence ψ onto w_ψ is an isomorphism of* Con(L) *and* $C(L)$.

3. PRINCIPLES OF IMPLEMENTATION

3.1. Canonical Forms and Tests of Identities

Every formula in the language of OMLs containing two variables (without loss of generality, a, b) can be associated to a unique element of the free OML F(a, b). (From now on, we identify formulas in a, b with the corresponding elements of F(a, b).) This allows to simplify such formulas to a canonical form. The

[6] The style file can be downloaded from ftp://math.feld.cvut.cz/pub/navara/foml2.sty.

program described here[7] expresses the elements of $\mathrm{F}(a, b)$ as joins of atoms. The finiteness of the OML $\mathrm{F}(a, b)$ allowed to implement all basic OML operations on $\mathrm{F}(a, b)$ and perform them automatically. (For the convenience of the user, all possible binary OML operations are implemented in our program.)

Once we have a (semantically) unique representation of all formulas in two variables, we can easily compare them (test identities) by just comparing the canonical forms. Also some relations on OMLs may be expressed as identities, e.g., the ordering,

$$a \leq b \quad \text{iff} \quad a = a \wedge b,$$

and commutation,

$$a \,\mathrm{C}\, b \quad \text{iff} \quad a = (a \wedge b) \vee (a \wedge b').$$

These relations are implemented in the program.

Let I be an identity of the form $x = y$, where $x, y \in \mathrm{F}(a, b)$ are formulas in variables a, b. Without loss of generality, I can be transformed to an equivalent identity of the form $u = 0$. There may be more elements $u \in \mathrm{F}(a, b)$ with this property; one of them is $d(x, y) = (x \vee y) \wedge (x' \vee y')$. (The operation d is one of possible symmetric differences in OMLs, see Dorfer *et al.*, 1996; the corresponding Beran's number is 89.) The identity I holds in $\mathrm{F}(a, b)$ (and hence in all OMLs) iff $d(x, y) = 0$. This can be easily checked by a program.

3.2. Congruences and Identities Satisfied in Quotient Algebras

Let I be an identity of the form $x = y$, where $x, y \in \mathrm{F}(a, b)$. If $d(x, y) \neq 0$, then I does not always hold. Besides the negative answer to its verification in general, we may also specify the "domain" of validity of I. This can be expressed in the terms of the least congruence such that I holds in the respective quotient algebra. We shall need the following fact:

Proposition 3.1. (Kalmbach, 1983, Section 6, Theorem 6) *Let L be an OML, ψ be a congruence on L, and $s, t \in L$. Then*

$$(s, t) \in \psi \quad \text{iff} \quad (s \vee t) \wedge (s' \vee t') \in \ker \psi.$$

Proposition 3.2. *Let I be an identity of the form $x = y$, $x, y \in \mathrm{F}(a, b)$. Then there exists the least congruence on $\mathrm{F}(a, b)$, denoted by φ_I, such that I holds in the quotient algebra $\mathrm{F}(a, b)/\varphi_I$. Moreover, $\ker \varphi_I = [0, v_I]$, where $v_I \in C(\mathrm{F}(a, b))$, and I is equivalent to the identity $v_I = 0$.*

[7] http://www.mat.savba.sk/∼hycko/oml.

Proof: For a congruence ψ, the identity I holds in the quotient algebra $\mathrm{F}(a, b)/\psi$ iff $(x, y) \in \psi$. The least congruence containing (x, y) is φ_I. Due to the finiteness of $\mathrm{F}(a, b)$, the kernel of φ_I is a principal ideal. The greatest elements of kernels of congruences in finite OMLs are always central (Proposition 2.4). □

The element v_I from Proposition 3.2 is also computed by our program. If we ask only of the validity of I, YES/NO answer is given and the value of v_I is available on request as an additional information. However, it plays an essential role in tests of implications, as we shall show in the sequel.

The program presented in Megill and Pavičić (2001) does what was described so far. Then it has been generalized in the direction particularly useful for the study of quantum implications. Here we develop another generalization which allows us to test more complex statements.

3.3. Tests of Implications

Let us now consider an implication between two identities.

Theorem 3.3. *Let I, J be identities of formulas in variables a, b. Then the following are equivalent:*

(1) *the implication $I \Longrightarrow J$ holds in all OMLs,*
(2) $\varphi_I \supseteq \varphi_J$,
(3) $v_I \geq v_J$,

where φ_I, φ_J are the congruences and v_I, v_J the central elements of $\mathrm{F}(a, b)$ *corresponding to I, J in the sense of Proposition 3.2.*

Proof: Conditions (2) and (3) are obviously equivalent, we shall prove their equivalence with (1).

Condition (1) holds in all OMLs iff J holds in all OMLs satisfying I. Then the sub-OML $G(a, b)$ generated by a, b is an image of the free OML $\mathrm{F}_I(a, b)$ with generators a, b satisfying I (thus a, b are *not free* generators of $\mathrm{F}_I(a, b)$). The OML $\mathrm{F}_I(a, b)$ is isomorphic to the quotient algebra $\mathrm{F}(a, b)/\varphi_I$ of the free OML $\mathrm{F}(a, b)$ with two *free* generators. Thus if J holds in $\mathrm{F}(a, b)/\varphi_I$, it holds also in $G(a, b)$ and in any OML satisfying I. We have obtained a sufficient condition $\varphi_I \supseteq \varphi_J$ which is equivalent to $v_I \geq v_J$.

To prove the necessity, assume that $v_I \not\geq v_J$, i.e., $\varphi_I \not\supseteq \varphi_J$. This means that J does not hold in $\mathrm{F}(a, b)/\varphi_I$. Then $\mathrm{F}(a, b)/\varphi_I$ itself is a counterexample on which the implication $I \Longrightarrow J$ does not hold. The proof is complete. □

Theorem 3.3 allows us to test implications between identities.

A conjunction of identities $I_1, \ldots, I_n$ is equivalent to a single identity. Indeed, $I_1, \ldots, I_n$, may be expressed in the forms $u_1 = 0, \ldots, u_n = 0$, and their conjunction is equivalent to the identity $u_1 \vee \cdots \vee u_n = 0$. Thus our tool allows to test conjunctions of OML identities in two variables and, by Theorem 3.3, also implications between conjunctions of identities, i.e., of the form $(I_1 \text{ AND} \cdots \text{AND } I_n) \Longrightarrow (J_1 \text{ AND} \cdots \text{AND } J_n)$. We go even further.

Note 3.4. We denote the classical logical conjunction and disjunction by AND and OR in order to distinguish them from the lattice operations in OMLs.

Theorem 3.5. *Let $n \in \mathbb{N}$ and let $I_1, \ldots, I_n, J_1, \ldots, J_n$ be identities of formulas in variables a, b. Then the following are equivalent:*

(1) *the conjunction of implications $(I_1 \Longrightarrow J_1) \text{ AND} \cdots \text{AND} (I_n \Longrightarrow J_n)$ holds in all OMLs,*
(2) $\forall i \in \{1, \ldots, n\} : \varphi_{I_i} \supseteq \varphi_{J_i}$,
(3) $\forall i \in \{1, \ldots, n\} : v_{I_i} \geq v_{J_i}$.

Proof: Again, conditions (2) and (3) are trivially equivalent and they are sufficient for (1).

To prove the necessity, assume that $\varphi_{I_i} \not\supseteq \varphi_{J_i}$ for some i. Then J_i does not hold in $\mathrm{F}(a, b)/\varphi_{I_i}$. Hence $\mathrm{F}(a, b)/\varphi_{I_i}$ itself is a counterexample on which the implication $I_i \Longrightarrow J_i$ does not hold. □

As the equivalence is a conjunction of two implications, it can be tested analogously:

Corollary 3.6. *Let I, J be identities of formulas in variables a, b. Then the following are equivalent:*

(1) *the equivalence $I \iff J$ holds in all OMLs,*
(2) $\varphi_I = \varphi_J$,
(3) $v_I = v_J$.

Corollary 3.7. *Let $n \in \mathbb{N}$ and let $I_1, \ldots, I_n, J$ be identities of formulas in variables a, b. Then the following are equivalent:*

(1) *the implication $(I_1 \text{ OR} \cdots \text{OR } I_n) \Longrightarrow J$ holds in all OMLs,*
(2) $\forall i \in \{1, \ldots, n\} : \varphi_{I_i} \supseteq \varphi_J$,
(3) $\forall i \in \{1, \ldots, n\} : v_{I_i} \geq v_J$.

Proof: From the classical logic, $(I_1 \text{ OR} \cdots \text{OR } I_n) \Longrightarrow J$ is equivalent to the conjunction of implications $(I_1 \Longrightarrow J) \text{ AND} \cdots \text{AND} (I_n \Longrightarrow J)$ and we apply Theorem 3.5. □

Thus we may allow a disjunction on the left-hand side of an implication. Due to the distributivity of classical logical operations AND, OR, the latter corollary allows also to test implications of the form $P(I_1, \ldots, I_n) \Longrightarrow J_1$ AND $\cdots$ AND J_n, where P is any lattice polynomial (using AND, OR) whose entries are identities in variables a, b. Obviously, also a conjunction of such implications can be tested by this technique and it is accepted by our program. However, this method does not allow to test general statements containing disjunctions and negations.

Example 3.8. Let us consider the disjunction I OR J, where I, J are the identities $a \wedge b = 0$, $a \wedge b' = 0$. Then $v_I = a \wedge b$, $v_J = a \wedge b'$; the least (central) element below v_I, v_J is 0. However, I OR J does not hold.

3.4. Extension to Finite Products of F(a, b)

As the validity of identities is preserved by direct products, our tool can be extended to finite direct products of copies of F(a, b) by applying the same principle to each copy of F(a, b) separately. This allows to make conclusions about general OMLs only in case that some free OML can be expressed as such a product. This is a very restrictive assumption. However, it is satisfied for a particular free orthomodular lattice, F$(a, b, c_1, \ldots, c_n)$, which has $n + 2$ generators $a, b, c_1, \ldots, c_n$ such that each c_i commutes with all other generators (the generators are not free). In Navara (1997), it is shown that F$(a, b, c_1, \ldots, c_n)$ is isomorphic to the direct product of 2^n copies of F(a, b). Its operations and properties are defined componentwise. The element x is central in F$(a, b, c_1, \ldots, c_n)$ iff it is central in each component F(a, b) in the direct product representation.

Our program allows to decide the validity of statements (restricted as in the previous section) in F$(a, b, c_1, \ldots, c_n)$ for $n \leq 9$. The results are applicable to any OML formula in variables $a, b, c_1, \ldots, c_n$ in any OML in which $\{a, b\}$ is the only noncommuting pair among $\{a, b, c_1, \ldots, c_n\}$. This allows to automatically check, e.g., the Foulis–Holland theorem and many other results using more than two variables.

Example 3.9. The following theorem is proved in Beran (1984, VII.7 Corollary 7.7): Let $\dot{\vee}$, $\underset{\bullet}{\wedge}$ denote the operations of *skew join* and *skew meet*, $x \dot{\vee} y = (x \wedge y') \vee y$, $x \underset{\bullet}{\wedge} y = (x \vee y') \wedge y$. If c commutes with both a and b, then the following conditions are equivalent:

(I) $a \wedge (c \vee b) \leq a \dot{\vee} b$,
(J) $a \underset{\bullet}{\wedge} b \leq a \vee (c' \wedge b)$,
(K) $a \underset{\bullet}{\wedge} (c \dot{\vee} b) = (a \underset{\bullet}{\wedge} c) \dot{\vee} (a \underset{\bullet}{\wedge} b)$ (distributivity of skew operations).

Our program allows to verify this theorem using a few lines of code. We may check a conjunction of equivalences, e.g. $(I \iff J)$ AND $(J \iff K)$, or use the standard cyclic implication method:

$$(I \Longrightarrow J)\,\text{AND}\,(J \Longrightarrow K)\,\text{AND}\,(K \Longrightarrow I)\,.$$

4. CONCLUSION AND LIMITATIONS OF APPLICABILITY

Free orthomodular lattices with more than two *free* generators are *infinite* Kalmbach (1983). Thus the possibility of further extension of our technique is very limited. Focusing is usually used for at least three variables, but it also uses an assumption on compatibility of the variables. While our technique allows just one noncommuting pair, focusing admits many *disjoint* pairs of noncommuting elements.

The simplest case in which focusing can be used and our technique is not applicable is the following: Let us assume a formula in four variables, a_1, b_1, a_2, b_2, such that the only noncommuting pairs are $\{a_1, b_1\}$ and $\{a_2, b_2\}$. Then focusing can be applied to any formula which does not contain orthocomplements. Our technique is not applicable to the free OML $\mathrm{F}(a_1, b_1, a_2, b_2)$ with generators a_1, b_1, a_2, b_2. The problem is not only that there are many factors in the direct product decomposition of $\mathrm{F}(a_1, b_1, a_2, b_2)$; one of these factors is neither Boolean, nor isomorphic to MO2, but much more complex.

On the other hand, our program admits to use orthocomplements arbitrarily. For example, it allows to work with all formulas from the free OML $\mathrm{F}(a, b, c_1)$ (where c_1 commutes with both a and b). This OML has $96^2 = 9216$ elements. The focusing technique admits only formulas without orthocomplements, in this case 20 expressions. (These are the elements of the free *bounded distributive* lattice with three free generators: 0, 1, and the 18 elements of the free distributive lattice with three free generators.) Orthocomplements of some variables may be allowed provided that the same variable does not occur without the orthocomplementation; this gives eight possible choices of orthocomplements of three variables and the total of at most $2 + 8 \times 18 = 146$ elements, much less than the formulas accepted by our program. Thus these techniques are incomparable and they both may simplify the work with formulas and statements in OMLs.

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